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Math441 HW8
October 26, 2004

6.4 #1

$$\frac{d^2y}{dt^2} + y = f(t) \qquad y(0) = 0,\ y'(0) = 1 \qquad f(t) = \begin{cases} 1, & 0 \le t < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \le t \end{cases}$$

$$\therefore f(t) = u_0(t) - u_{\frac{\pi}{2}}(t)$$

$$\text{L.T.:}\quad s^2Y - sy(0) - y'(0) + Y = \frac{e^{0s}}{s} - \frac{e^{-\frac{\pi}{2}s}}{s} = \frac{1}{s} - \frac{e^{-\frac{\pi}{2}s}}{s}$$

$$s^2Y - 1 + Y =$$

$$(s^2+1)Y = 1 + \frac{1}{s} - \frac{e^{-\frac{\pi}{2}s}}{s}$$

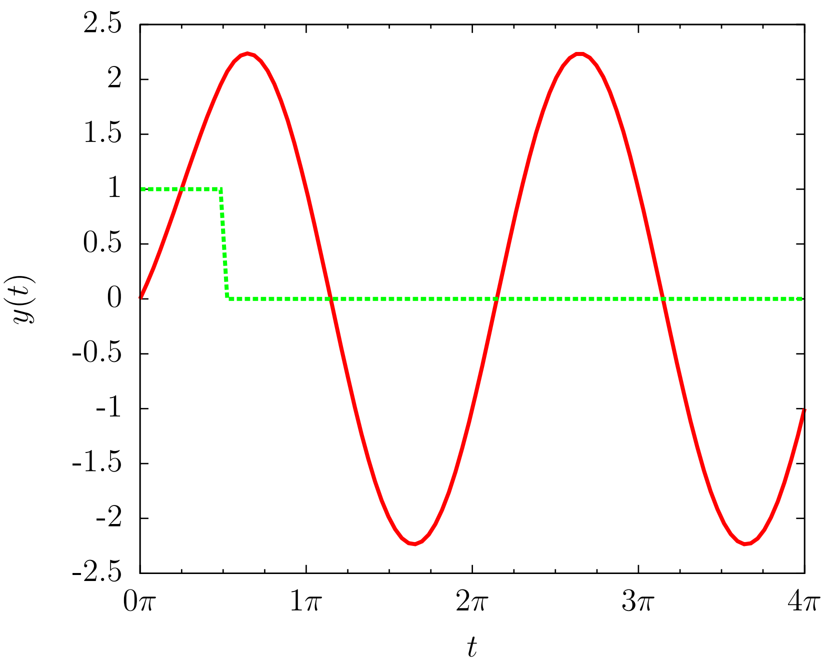
$$Y = \frac{1}{(s^2+1)} + \frac{1}{s(s^2+1)} - \frac{e^{-\frac{\pi}{2}s}}{s(s^2+1)}$$

$$\text{Partial Fractions:}\quad = \frac{1}{(s^2+1)} + \frac{1}{s} - \frac{s}{s^2+1} - \frac{e^{-\frac{\pi}{2}s}}{s} + \frac{se^{-\frac{\pi}{2}s}}{s^2+1}$$

$$= \frac{1}{(s^2+1)} + \frac{1}{s} - \frac{s}{s^2+1} - e^{-\frac{\pi}{2}s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$$

$$y(t) = \sin(t) + 1 - \cos(t) - u_{\frac{\pi}{2}}\left(1 - \cos\left(t - \frac{\pi}{2}\right)\right)$$

$$= \sin(t) + 1 - \cos(t) - u_{\frac{\pi}{2}}(1 - \sin(t))$$



6.5 #25 (a)

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = f(t) \qquad \begin{matrix} y(0) = 0 \\ y'(0) = 0 \end{matrix}$$

$$\begin{aligned}
r^2 + 2r + 2 &= 0 \\
r^2 + 2r &= -2 \\
r^2 + 2r + 1 &= -2 + 1 \\
(r+1)^2 &= -1 \\
r + 1 &= \pm\imath \\
r &= -1 \pm \imath
\end{aligned}$$

$$y_1 = e^{t(-1+\imath)} = e^{t(\imath-1)},\ y_2 = e^{t(-1-\imath)} = e^{-t(1+\imath)}$$

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = f(t) \end{cases}$$

$$u_1' = \frac{e^{(1-\imath)t}}{2\imath}f(t),\ u_2' = -\frac{e^{(1+\imath)t}}{2\imath}f(t)$$

$$y = u_1y_1 + u_2y_2$$

Rewrite $u'_{1,2}$ in terms of τ:

$$\begin{aligned}
&= \int_0^t u_1'(\tau)y_1(t)d\tau + \int_0^t u_2'(\tau)y_2(t)d\tau \\
&= \int_0^t u_1'(\tau)y_1(t) + u_2'(\tau)y_2(t)\ d\tau \\
&= \int_0^t \frac{e^{(1-\imath)\tau}}{2\imath}f(\tau)e^{t(\imath-1)} - \frac{e^{(1+\imath)\tau}}{2\imath}f(\tau)e^{-t(1+\imath)}\ d\tau \\
&= \int_0^t \frac{f(\tau)}{2\imath}\left(e^{(1-\imath)\tau}e^{t(\imath-1)} - e^{(1+\imath)\tau}e^{-t(1+\imath)}\right) d\tau \\
&= \int_0^t \frac{f(\tau)}{2\imath}\left(e^{(1-\imath)\tau}e^{-t(1-\imath)} - e^{(1+\imath)\tau}e^{-t(1+\imath)}\right) d\tau \\
&= \int_0^t \frac{f(\tau)}{2\imath}\left(e^{(1-\imath)(\tau-t)} - e^{(1+\imath)(\tau-t)}\right) d\tau \\
&= \int_0^t \frac{f(\tau)}{2\imath}\left(e^{(\tau-t)}e^{-\imath(\tau-t)} - e^{(\tau-t)}e^{\imath(\tau-t)}\right) d\tau \\
&= \int_0^t f(\tau)e^{(\tau-t)}\left(\frac{e^{-\imath(\tau-t)} - e^{\imath(\tau-t)}}{2\imath}\right) d\tau \\
&= \int_0^t f(\tau)e^{(\tau-t)}\left(\frac{e^{\imath(t-\tau)} - e^{-\imath(t-\tau)}}{2\imath}\right) d\tau \\
&= \int_0^t e^{-(t-\tau)}f(\tau)\sin(t-\tau)d\tau
\end{aligned}$$

(b)

$$\begin{aligned}
y &= \int_0^t e^{-(t-\tau)} f(\tau)\sin(t-\tau)d\tau \quad \text{Given: } f(t)=\delta(t-\pi)\\
&= \int_0^t e^{-(t-\tau)}\delta(\tau-\pi)\sin(t-\tau)d\tau\\
&= \int_0^t e^{-(t-\pi)}\delta(\tau-\pi)\sin(t-\pi)d\tau\\
&= e^{-(t-\pi)}\sin(t-\pi)\int_0^t \delta(\tau-\pi)d\tau\\
&= u_\pi(t)\ e^{-(t-\pi)}\sin(t-\pi)
\end{aligned}$$

(c)

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+2y = f(t) \qquad \begin{aligned} y(0)&=0\\ y'(0)&=0\end{aligned}$$

$$\begin{aligned}
\text{L.T.:}\quad s^2Y - sy(0) - y'(0) + 2sY - 2y(0) + 2Y &= F(t)\\
(s^2+2s+2)Y &= \delta(t-\pi)\\
Y &= \frac{e^{-\pi s}}{s^2+2s+2}\\
y &= \mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s^2+2s+2}\right]\\
&= \mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s^2+2s+1+1}\right]\\
&= \mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{(s+1)^2+1}\right]\\
&= e^{-t}\mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s^2+1}\right]\\
&= u_\pi(t)e^{-(t-\pi)}\sin(t-\pi)
\end{aligned}$$

6.6 #7

$$\begin{aligned}
f(t) &= \int_0^t \sin(t-\tau)\cos(\tau)\ d\tau\\
F(s) &= \frac{1}{s^2+1}\ \frac{s}{s^2+1} = \frac{s}{(s^2+1)^2}
\end{aligned}$$

6.6 #9

$$\begin{aligned}
F(s) &= \frac{s}{(s+1)(s^2+4)}\\
&= \frac{1}{s+1}\ \frac{s}{s^2+4}\\
f(t) &= \int_0^t e^{-(t-\tau)}\cos(2\tau)\ d\tau
\end{aligned}$$

6.6 #20

$$\begin{aligned}
\phi(t) + \int_0^t k(t-\xi)\phi(\xi)\,d\xi &= f(t) \\
\phi(s) + \phi(s)K(s) &= F(s) \\
\phi(s)(1+K(s)) &= F(s) \\
\phi(s) &= \frac{F(s)}{1+K(s)}
\end{aligned}$$